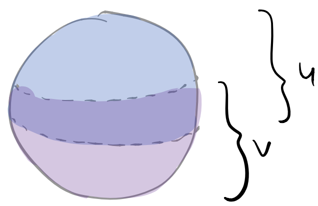
Intro to van-Kampen's Theorem

Question: If we know that $X = U \cup V$, U and V open, and we know how to calculate $\pi_1(U)$ and $\pi_1(V)$, how do we find $\pi_1(X)$?

e.g.



$S^2 = U \cup V$
and $\pi_1(U)$
and $\pi_1(V)$ are
trivial.

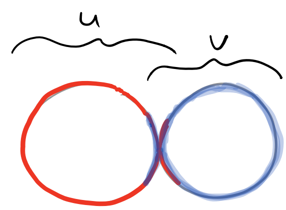


figure 8 space $= U \cup V$,
both have homotopy type
of S^1

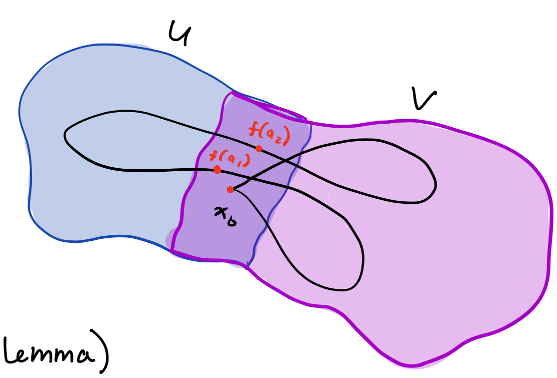
The Seifert-van Kampen Theorem, which we will see later, tells us exactly how to calculate $\pi_1(X)$ if it's the union of spaces like this. We'll first prove this weaker version of the theorem:

Theorem: Suppose $X = U \cup V$, where U and V are open, $U \cap V$ is path connected, and $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the inclusion maps. Then the images of

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

Pf: Let f be a path in X.
Subdivide $[0,1]$ into
$0 = a_0 < a_1 < \ldots < a_n = 1$ s.t. $f(a_i) \in U \cap V$
and $f([a_i, a_{i+1}]) \subseteq U$ or V. (Use Lebesgue # Lemma)



Let $f_i = f|_{[a_{i-1},\, a_i]}$, so $[f] = [f_1] * \dots * [f_n]$.

For each i, choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$.

($\alpha_0 = \alpha_n$ = constant path at x_0).

run α_i backwards

Set $g_i = \alpha_{i-1} * f_i * \overline{\alpha_i}$ $\quad$ g_i is a loop at x_0 in U or V.

Then $[g_1] * [g_2] * \dots * [g_n] = [f_1] * [f_2] * \dots * [f_n] = [f]$,

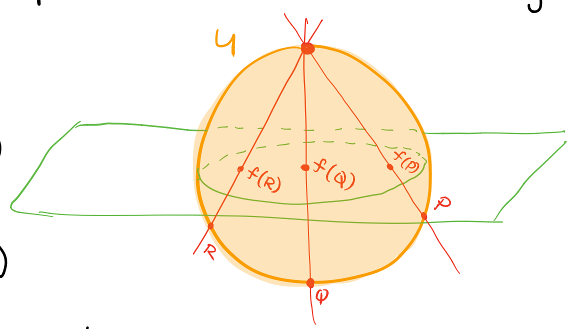
since $[\overline{\alpha_i}] * [\alpha_i] = [\text{constant path}]$. $\square$

Cor: If $X = U \cup V$, U and V open and simply connected and $U \cap V$ path connected, then X is simply connected.

Ex: Let $X = S^n$ $(n \geq 2)$ and $U = S^n - (0, 0, \dots, 1)$, $V = S^n - (0, 0, \dots, -1)$

North Pole $\quad$ South Pole

Then U and V are homeomorphic to $\mathbb{R}^n$ via the stereographic projection $f : U \to \mathbb{R}^n$



defined $f(x) = \dfrac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$

(change to + for V)

(exercise: check this is a homeo.)

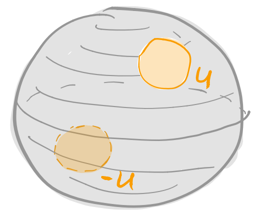
$U \cap V$ is thus homeomorphic to $\mathbb{R}^n$ minus a point, so it's path connected. $\Rightarrow$ S^n is simply connected.

Ex: Recall the quotient map from HW 8 $f : S^n \to S^n/\sim$ $(x \sim -x)$

We showed $S^n/\sim$ is homeomorphic to $\mathbb{R}P^n$.

In fact this is a covering map, since for $x \in S^n/\sim$, we can choose $y \in f^{-1}(x)$, and $\varepsilon < 1$ so that $U = B_\varepsilon(y) \cap S^n$ contains no antipodal points. Thus f maps U homeomorphically onto its image, and $f^{-1}(f(U)) = U \sqcup -U$, both homeomorphic to $f(U)$.

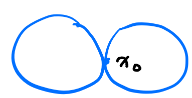
Let $x_0 \in \mathbb{RP}^n$ be a base point, and $n \geq 2$.
Since S^n is simply connected, the lifting correspondence



$$\varphi : \pi_1(\mathbb{RP}^n, x_0) \to \underset{\substack{\shortparallel \\ \text{2 antipodal} \\ \text{points}}}{f^{-1}(x_0)} \quad \text{is bijective.}$$

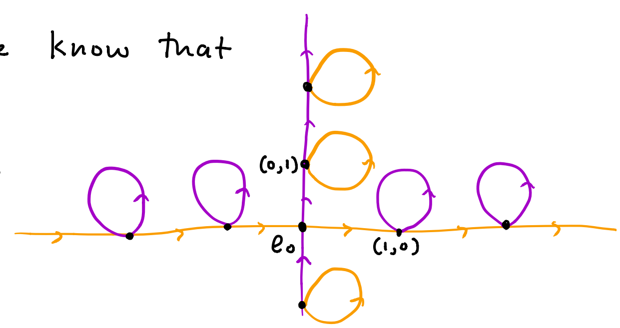
So $\pi_1(\mathbb{RP}^n, x_0)$ has two elements, so it's isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

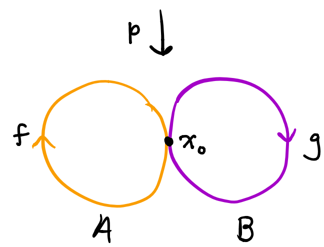
Ex: Let X be the figure-eight space



We can cover it w/ two open sets that have deformation retractions onto S^1. Thus, we know that

π_1 is generated by the images of two maps from $\mathbb{Z}$, but we don't know their relations.
Consider the covering map of X ⟶





Let e_0 be the basepoint of the covering space, x_0 the base point of X.

Let f and g be the paths that wrap once around A and B, respectively.

The lift $\widetilde{f*g}$ ends at $(1,0)$, whereas the lift $\widetilde{g*f}$ ends at $(0,1)$. Therefore, $[f]*[g] \neq [g]*[f]$, so $\pi_1(X, x_0)$ is not abelian. In fact, later we'll show that it is the free group generated by two elements.

i.e. the elements are of the form $[h_1]^{\alpha_1} * [h_2]^{\alpha_2} * \dots * [h_n]^{\alpha_n}$, $h_i \in \{f, g\}$ and $\alpha_i \in \mathbb{Z}$. We'll talk more about free groups later.